A GEOMETRICAL PROBLEM CONNECTED WITH THE CONTINUATION OF A POWER–SERIES

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GIVEN a power-series $P(x)$ with only one singular point A on the circumference of its circle of convergence.

Denote this circle by C_1, its center by M.

Let us suppose that all the other singular points of the analytic function defined by the element $P(x)$ are so situated as not to interfere with the continuations of $P(x)$ which are to be considered.

The following purely geometrical question arises: *How are the intermediate circles to be chosen in order to arrive again at a circle with center M, by a minimum number of continuations around the point A?*

To answer this question we consider the total area covered by all direct (first) continuations of $P(x)$. The boundary of this area is the envelope of the circles passing through A whose centers lie on the circumference of C_1. This envelope is a cardioid C_2. The boundary of the area covered by all the second continuations is the envelope C_3 of all circles through A whose centers lie on C_2. Continuing this process, we obtain a series of curves $C_2, C_3, \cdots, C_n$, where C_n is the boundary of the area covered by all $(n-1)$st continuations.

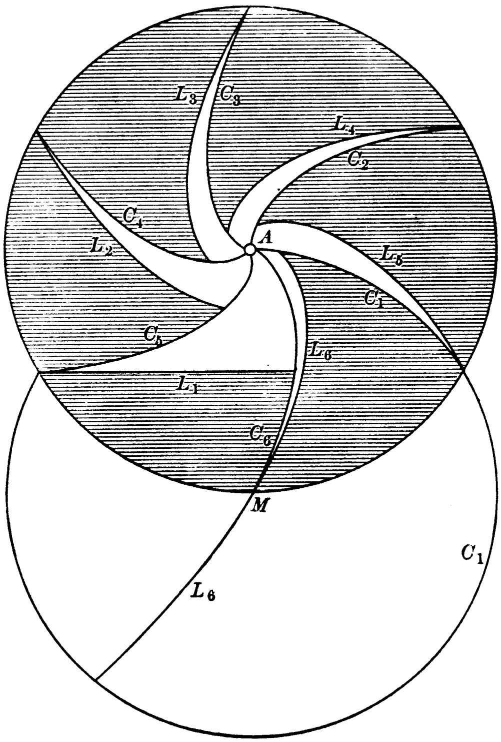
Counting radii vectores ρ from A and polar angles θ from AM, and taking the radius $AM = 1$, the equations of the successive curves appear in

the following simple form:

$$C_1 \text{ (given circle)}: \quad \rho = 2 \cos \theta;$$

$$C_2 \text{ (cardioid)}: \quad \rho = 4 \cos^2(\tfrac{1}{2} \theta);$$

$$C_n \text{ (sine-spiral)}: \quad \rho = 2^n \cos^n(\tfrac{1}{n} \theta).$$



These curves are so-called sine-spirals.* With regard to the curves C_n the following theorem can easily be proved:

Define a series of points $P_1, P_2, \cdots, P_n, \cdots$, where P_1 lies arbitrarily on C_1, and where every P_{n+1} is that point on C_{n+1} (for $n = 1, 2, 3, \cdots$)

* Compare Scheffers' article in the *Encyklopädie der mathematischen Wissenschaften*, vol. III, D 4, §§21–24.

in which C_{n+1} is touched by the circle with center P_n and radius P_nA. Then for the polar coordinates ρ_n and θ_n of P_n the following relations hold:

$$\rho_n = \rho_1^n, \quad \theta_n = n\theta_1.$$

Choosing $\rho_1 = 1$, we have $\theta_1 = \pi/3$ and every $\rho_n = 1$. Hence, a circle with radius 1 about A meets $C_1, C_2, \cdots, C_6$ in six points which form a regular hexagon. The curve C_6 passes, therefore, through M.

Six continuations at least are then required in order that the center M shall lie inside of the last one, and seven continuations at least are necessary to reach a circle with M as center.

In order that the sixth continuation shall contain the center M it is necessary that the centers of two consecutive continuations be always separated by one of the curves C_n.

In the adjoining figure the curves C_n are drawn only so far as is required for continuations around A in the positive sense (counter-clockwise), and only that part of them is shown in the figure which lies in a circle with radius 1 about the singular point A as center.

Denote now the center of the n^{th} continuation by T_n. If we wish T_7 to coincide with M, T_6 must lie below the straight line L_1 bisecting AM and perpendicular to it (see diagram). This defines for T_5 also a certain limiting curve L_2, the locus of the centers of circles passing through A and touching L_1. Likewise we obtain for T_4, T_3, T_2, T_1 the limiting curves L_3, L_4, L_5, L_6 respectively.

The equations of these curves are:

$$L_1 \text{ (straight line)}: \frac{1}{\rho} = 2\cos\theta;$$

$$L_2 \text{ (parabola)}: \frac{1}{\rho} = 4\cos^2(\tfrac{1}{2}\theta);$$

$$L_n \text{ (sine-spiral)}: \frac{1}{\rho} = 2^n\cos^n(\tfrac{1}{n}\theta).$$

These curves L_n are again sine-spirals.

Every L_n (for $n = 1, 2, \cdots, 6$) touches the corresponding C_{6-n} (where $C_0 = C_6$) on the unit-circle around A.

Hence, in our diagram, for continuations around A in the positive sense, none of the centers T must lie in the unshaded region of the circle with center A.

Thus we have reached the following result:

In order that the center T_7 *of the seventh continuation shall coincide with* M *it is necessary and sufficient that the center of the first continuation* T_1 *(for continuations around* A *in the positive sense) lie in the circle* C_1 *on the right side of* L_6 *(compare diagram), and that the centers* T_n *of any two consecutive continuations be separated not only by a curve* C*, but also by a curve* L*.*

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